

**Modeling and Control of Flexible Space Platforms
with Articulated Payloads**

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Abstract

The first steps in developing a methodology for spacecraft control-structure interaction (CSI) optimization are identification and classification of anticipated missions, and the development of tractable mathematical models in each mission class. A mathematical model of a generic large flexible space platform (LFSP) with multiple, independently pointed rigid payloads (representative of "Class II" CSI missions) is considered. The objective here is not to develop a general purpose numerical simulation, but rather to develop an analytically tractable mathematical model of such composite systems. The equations of motion for a single payload case are derived, and are linearized about zero steady-state. The resulting model is then extended to include multiple rigid payloads, yielding the desired analytical form. The mathematical models developed clearly show the internal inertial/elastic couplings, and are therefore suitable for analytical and numerical studies. A simple decentralized control law is proposed for fine pointing the payloads and LFSP attitude control, and simulation results are presented for an example problem. The decentralized controller is shown to be adequate for the example problem chosen, but does not, in general, guarantee stability. A centralized dissipative controller is then proposed, requiring a symmetric form of the composite system equations. Such a controller guarantees robust closed-loop stability despite unmodeled elastic dynamics and parameter uncertainties.

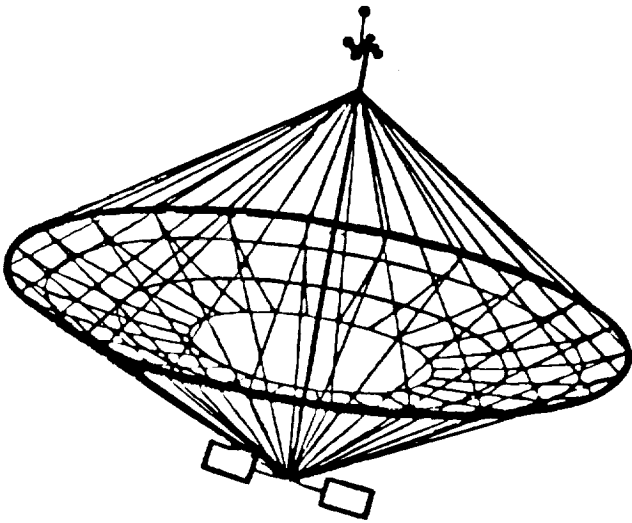
Outline

- CSI Mission Classification
- Mathematical Model of LFSP/ Single Payload
- Extension to Multi-Payload Systems
- Decentralized Control Law
- Simulation Results for Space Station/Articulated Payloads
- Centralized Dissipative Controller
 - guarantees closed-loop stability
 - requires symmetric form of eqns. of motion
- Concluding Remarks

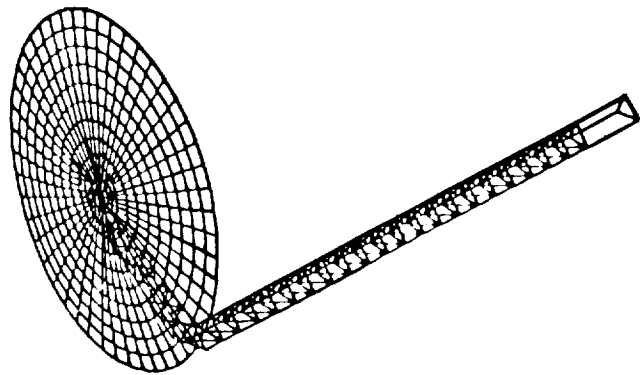
CSI Mission Classification

Anticipated missions involving control-structure interaction (CSI) can be broadly divided into four classes. Class I missions are those which require fine-pointing of the overall spacecraft as well as vibration suppression. There are no articulated substructures, and because of small elastic and rigid-body motion, the modeling and control problem is essentially linear. There may, however, be actuator and sensor nonlinearities. Examples of this mission class include large space antenna concepts such as the hoop-column and wrap-rib antennas. The controller must satisfy the performance specifications, which include fine-pointing (rigid plus flexible rotational motion), reflector surface distortion, and defocus errors caused by feed and base elastic motion. The control system must be capable of performing in the presence of significant elastic motion.

- **Class I :** Single-Body Flexible Spacecraft (Linear)
 - fine-pointing and vibration suppression
 - surface shape distortion and defocus errors



hoop-column antenna concept



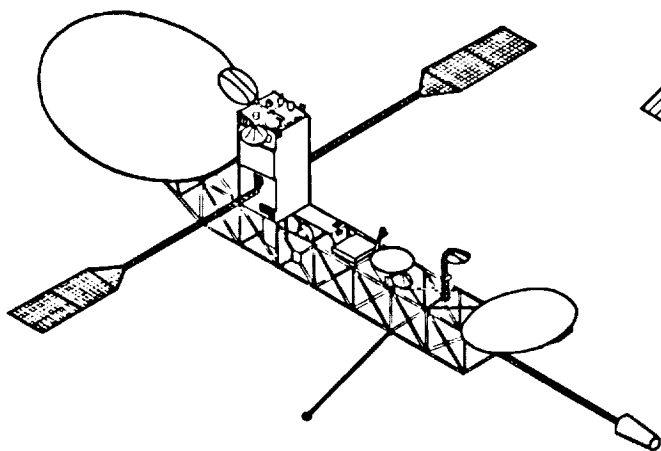
wrap-rib antenna concept

CSI Mission Classification (cont.)

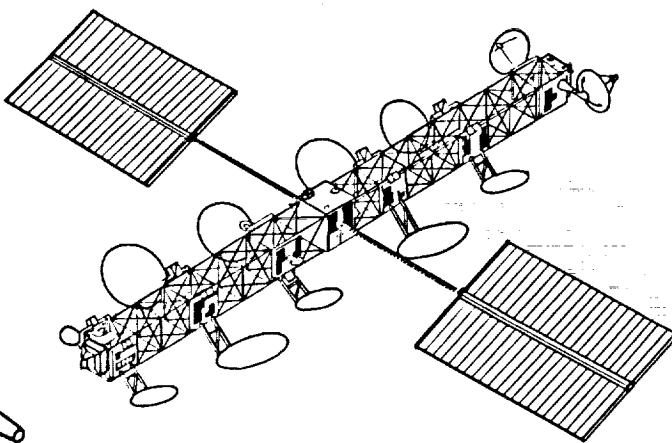
Class II missions represent an extension of Class I missions with articulated payloads mounted on the flexible spacecraft. This important class of missions includes many types of low-earth and geosynchronous platforms. The composite system consists of a large flexible space platform (LFSP) to which a number of rigid or elastic appendages are mounted. The objectives of the controller are precision attitude control of the spacecraft, fine-pointing of each of its payloads, and vibration suppression. The problem is still linear, but the elastic and rigid-body motions between bodies are coupled.

- **Class II : Flexible Spacecraft with Articulated Appendages (Linear)**

- includes rigid and elastic appendages (payloads)
- fine-pointing of all rigid and elastic components
- vibration suppression to improve payload performance



Earth Observation System (EOS)



geosynchronous platform

CSI Mission Classification (cont.)

Class III and IV missions are essentially the nonlinear counterparts to the Class I and II missions previously discussed. Class III missions will require large angle maneuvers of the entire structure (without articulated appendages) while simultaneously minimizing the effects of elastic motion induced by nonlinear internal couplings and external disturbances. Although this problem has been studied numerically for specific flexible spacecraft, generic models do not exist.

Class IV missions have the inherent difficulty of the nonlinear Class III problems, but include the additional complexity of the articulated appendages of their linear Class II counterparts. Such missions may require large-angle maneuvering of the flexible structure while simultaneously and independently pointing various payloads to their own respective targets in the presence of elastic motion. There is rigid/elastic coupling between all of the various components. Furthermore, payloads may be repositioned, or the spacecraft reconfigured by commanded motions of the various appendages. Examples of this mission class include LFSPs with articulated payloads and robotic arms allowing translational degrees of freedom.

The purpose of this paper is to develop a generic mathematical model and control laws for an LFSP with multiple articulated payloads representative of Class II missions suited to CSI optimization.

- **Class III :** Single-Body Flexible Spacecraft (Nonlinear)
 - Precision large angle maneuvers for retargeting or tracking
 - Vibration suppression during maneuvers
- **Class IV :** Flexible Spacecraft with Articulated Appendages (Nonlinear)
 - Large angle rotational motion of spacecraft and appendages
 - Robotic manipulators allowing spacecraft reconfiguration

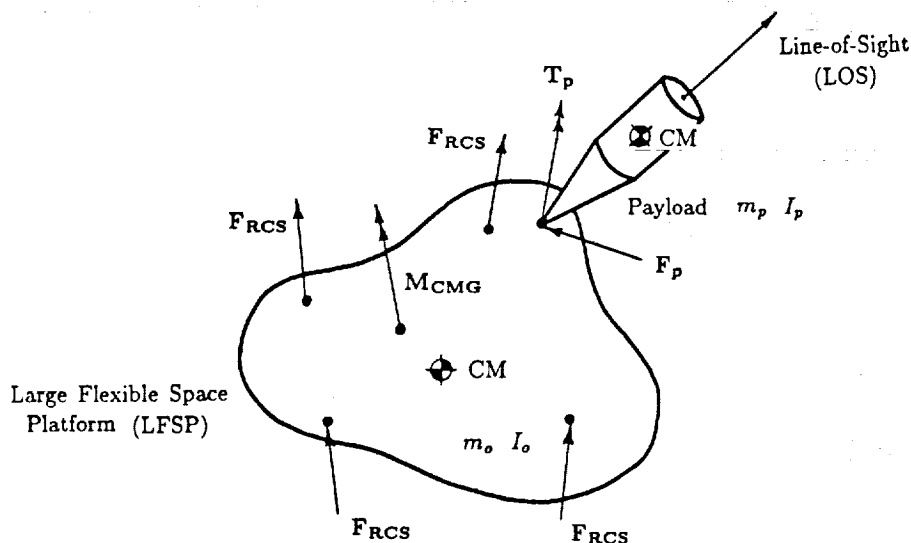
Mathematical Model for LFSP/Single Payload

In developing the equations of motion, the single payload composite system will first be described in terms of its various coordinate systems, position vectors, etc. The equations of motion for each component comprising the composite system, including LFSP rigid and elastic motion as well as payload rigid-body motion, will then be presented. By removing the constraint force between the LFSP and payload, these equations will then be coupled, and presented in matrix form. This generic model will then be extended to the multi-payload case.

A large flexible central body with a single attached articulated rigid payload is considered first. The mass/inertia properties of both bodies are assumed known. The modal solution (frequencies and displacement vectors) for n modes of the flexible body (LFSP) is also assumed known. This modal solution is usually obtained using finite element codes such as NASTRAN or EAL, wherein the effects of the payloads are taken into account by representing them as point masses and inertias on the LFSP model. The present analysis requires the values of LFSP mode shapes and mode slopes at the points of application of forces and torques, respectively. Both are required at the payload attachment point. The attitude control system for the LFSP is assumed to utilize reaction control system (RCS) thrusters, and control moment gyro (CMG) torques. The payload pointing mechanism is idealized as a three-axis point torque acting at the attachment gimbal. The objective is to control the payload line-of-sight and LFSP attitude in the presence of elastic motion.

Overall Configuration

- LFSP with single rigid payload
- Control objectives:
 - attitude control of LFSP
 - fine-pointing of payload
- Data required for model
 - payload and LFSP masses and inertias
 - modal frequencies
 - mode shapes for forces (e.g.: RCS thrusters)
 - mode slopes for moments (e.g.: CMG moments)

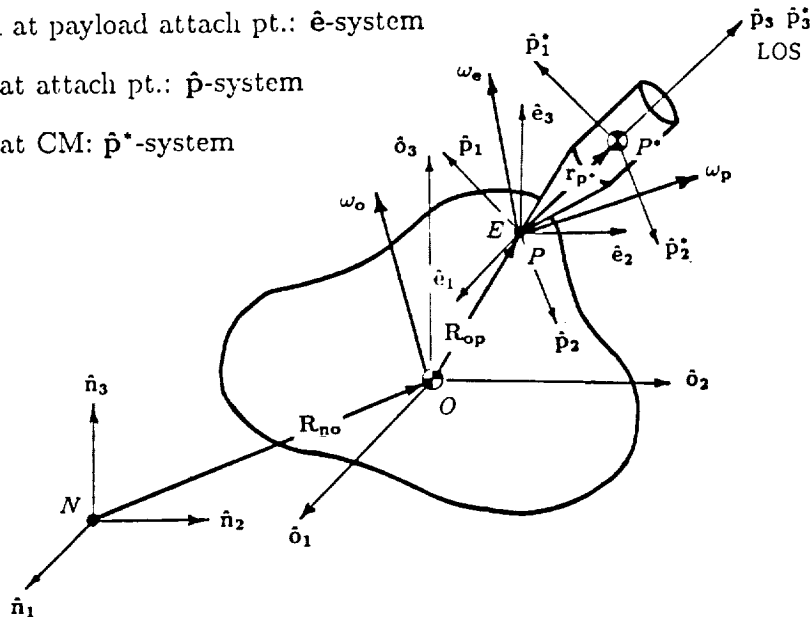


Coordinate Systems

The LFSP has two coordinate systems associated with it. One, designated the \hat{o} -system, is fixed to the nominal center of mass (CM), and in conjunction with the nominally zero Euler angles $(\phi_o, \theta_o, \psi_o)$, describes the rigid-body orientation. The other coordinate system, designated the \hat{e} -system, is fixed to the payload attach point ("point P") on the LFSP, thus rotating and translating with the rigid and elastic LFSP motion at that point. This system is nominally parallel to the \hat{o} -system axes. The orientation of the elastic axis system (\hat{e} -system) with respect to the \hat{o} -system axes is described by Euler angles $(\phi_e, \theta_e, \psi_e)$ (i.e., the rotational elastic displacements), which are nominally zero. The origin of the LFSP \hat{o} -system is located in the reference inertial coordinate system (denoted the \hat{n} -system), by the position vector $\mathbf{R}_{no} = X\hat{n}_1 + Y\hat{n}_2 + Z\hat{n}_3$. The payload attachment point P for the undeformed spacecraft is located from the LFSP CM, the \hat{o} -system origin, by position vector $\mathbf{R}_{op} = x_p\hat{o}_1 + y_p\hat{o}_2 + z_p\hat{o}_3$.

The rigid payload is described by a body-fixed \hat{p} -system originating from the attachment point, but fixed to the payload. The \hat{e} - and \hat{p} -system origins are coincident. Orientation with respect to the elastic axes is described by Euler angles $(\phi_p, \theta_p, \psi_p)$. These are the payload pointing mechanism "gimbal angles". The desired nominal orientation is defined by user-specified gimbal angles. Thus the angular orientation of the payload may be determined from the orientation of the \hat{p} -system with respect to the \hat{e} -system; the \hat{e} -system with respect to the \hat{o} -system (due to elastic deformation); and the orientation in inertial space of the \hat{o} -system. Expressing a vector in terms of components in the various coordinate systems requires transformations using a 1-2-3 Euler sequence of rotations. The angular velocity vectors (body rates) of the various coordinate frames are ω_o , ω_e , and ω_p for the LFSP rigid frame, LFSP elastic frame, and the payload frame, respectively. The body rates must be transformed into Euler rates prior to linearization.

- inertial reference: \hat{n} -system
- LFSP (rigid) body axes: \hat{o} -system
- elastic system at payload attach pt.: \hat{e} -system
- payload axes at attach pt.: \hat{p} -system
- payload axes at CM: \hat{p}^* -system



Nonlinear Model

The nonlinear equations of motion for the composite system are developed next. The bodies are each assumed to have constant mass and mass distribution, and to possess only rotational freedom with respect to each other. All terms in the equations represent inertial quantities which are to be expressed in the \hat{o} -system, nominally parallel to the inertial frame. It should be noted that " $\frac{d}{dt}$ " denotes differentiation with respect to the inertial frame, and so angular velocity cross-products appear, whereas an overdot " $\dot{}$ " indicates a local time derivative inside a particular frame. Appropriate coordinate transformations in terms of Euler angles, using a 1-2-3 sequence, are required.

First consider the LFSP equations of motion. Beginning with the translational equation of motion, writing Newton's law gives equation (1). Here m_o is the LFSP mass, F_p is the internal constraint (reaction) force exerted on the LFSP by the payload, and there are n_F external forces F_{oi} acting on the body (e.g., RCS thrusters).

Next the nonlinear rotational equation of motion of the LFSP is given by equation (2). I_o is the LFSP inertia matrix about the CM, T_p is the pointing system reaction torque, and there are n_M moments M_{oi} acting on the LFSP (e.g., CMG moments). The position vector $R_{op} = x_p \hat{o}_1 + y_p \hat{o}_2 + z_p \hat{o}_3$ locates the payload attachment point, and the position vector $r_{oi} = x_{oi} \hat{o}_1 + y_{oi} \hat{o}_2 + z_{oi} \hat{o}_3$ locates the i^{th} external force F_{oi} acting on the LFSP. Here the "prime" notation with R'_{op} and r'_{oi} indicates the (3×3) matrix cross-product form.

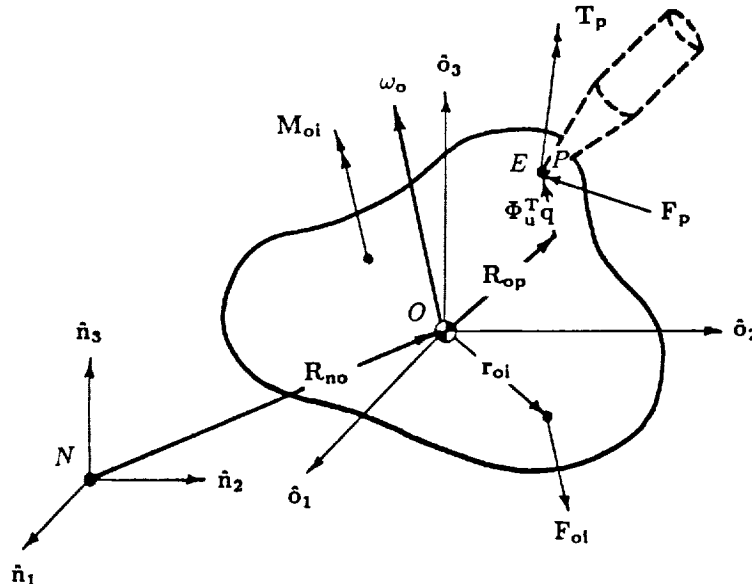
LFSP Eqns. of Motion

- LFSP translation:

$$m_o \ddot{R}_{no} = -F_p + \sum_{i=1}^{n_F} F_{oi} \quad (1)$$

- LFSP rotation:

$$I_o \dot{\omega}_o + \omega_o \times I_o \omega_o = -R_{op} \times F_p - T_p + \sum_{i=1}^{n_M} M_{oi} + \sum_{i=1}^{n_F} r_{oi} \times F_{oi} \quad (2)$$

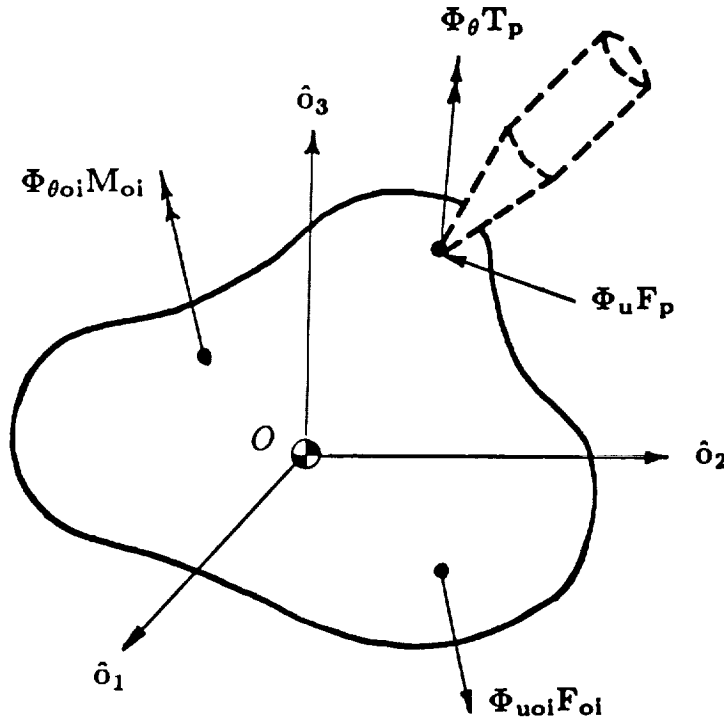


Nonlinear Model (cont.)

The LFSP elastic motion is assumed to be decoupled from the LFSP rigid-body motion (valid for small motion), so that inertial/elastic coupling terms arise from payload interactions with the platform at the attachment point. Thus the elastic motion of the LFSP may be described by equation (3), the modal equation of motion. The $(n \times 1)$ vector \mathbf{q} contains the modal coordinates, where n is the number of modes used to describe the elastic motion. The modal coordinates and their corresponding time derivatives will comprise $2n$ components of the system state vector. The matrix \mathbf{D} is a diagonal matrix with terms $2\zeta_i\omega_i$, where ζ_i and ω_i are the damping ratio and frequency, respectively, for elastic mode i . Matrix $\mathbf{\Lambda}$ is a diagonal matrix of ω_i^2 . The Φ 's consist of mode slopes at the point where a moment is applied, or mode shapes at force application points. The modal matrices Φ are dimensioned $(n \times 3)$. The right-hand side of the expression represents modal forces and moments.

LFSP Elastic Motion

$$\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{\Lambda}\mathbf{q} = -\Phi_u\mathbf{F}_p - \Phi_\theta\mathbf{T}_p + \sum_{i=1}^{n_M} \Phi_{\theta oi}M_{oi} + \sum_{i=1}^{n_F} \Phi_{u oi}\mathbf{F}_{oi} \quad (3)$$



Nonlinear Model (cont.)

Next consider the payload equations. The payload CM location, point P^* , for the deformed spacecraft with respect to the inertial reference is given by vector \mathbf{R}_{np^*} in (4) below. The acceleration of the payload CM is sought. Consider time derivatives of the position vector \mathbf{R}_{np^*} with respect to the inertial frame, that is, take " $\frac{d}{dt}$ " of the right-hand side of \mathbf{R}_{np^*} . This gives the velocity of the payload CM, $\dot{\mathbf{R}}_{np^*}$. Taking another time derivative, again including terms due to the relative rotations between coordinate systems, yields the inertial acceleration of the payload CM, $\ddot{\mathbf{R}}_{np^*}$. The only external force acting on the payload is the constraint force at the attachment point. Writing Newton's law gives the nonlinear translational equation of motion, equation (5).

Lastly, the nonlinear payload rotational equation of motion (about the payload CM) is given by equation (6). \mathbf{I}_p is the payload centroidal inertia matrix for the $\hat{\mathbf{p}}^*$ -system, and \mathbf{M}_p is the total moment about the payload CM. Taking appropriate time derivatives and characterizing the moment \mathbf{M}_p acting on the payload results in equation (7).

Nonlinear Payload Equations of Motion

- Payload CM translation:

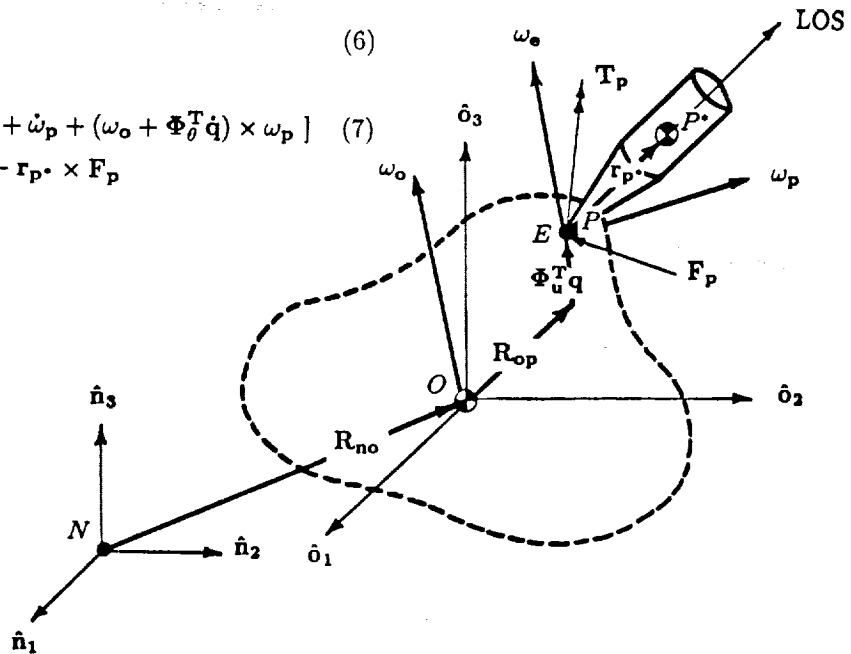
$$\mathbf{R}_{np^*} = \mathbf{R}_{no} + \mathbf{R}_{op} + \Phi_u^T \mathbf{q} + \mathbf{r}_{p^*} \quad (4)$$

$$m_p \ddot{\mathbf{R}}_{np^*} = \mathbf{F}_p \quad (5)$$

- Payload rotation:

$$\mathbf{I}_p \frac{d}{dt}(\omega_o + \omega_e + \omega_p) = \mathbf{M}_p \quad (6)$$

$$\mathbf{I}_p [\dot{\omega}_o + \Phi_\theta^T \ddot{\mathbf{q}} + \omega_o \times \Phi_\theta^T \dot{\mathbf{q}} + \dot{\omega}_p + (\omega_o + \Phi_\theta^T \dot{\mathbf{q}}) \times \omega_p] = \mathbf{T}_p - \mathbf{r}_{p^*} \times \mathbf{F}_p \quad (7)$$



Linearized Model

Consider now transformation of all the inertial quantities to $\hat{\mathbf{o}}$ -system components, linearization of the nonlinear equations, and the introduction of perturbations (to nominal values). Note that the $\hat{\mathbf{o}}$ -system is nominally parallel to the inertial frame. Upon linearization, system variables may be thought of as being comprised of a nominal value plus a perturbation quantity, for example some variable $y = y_0 + \delta y$. The prefix “ δ ” will be used to denote departure values from the equilibrium (nominal) state. The nominal attitude of the LFSP is $\mathbf{a}_o = (\phi_o, \theta_o, \psi_o)^T = \mathbf{0}$, the nominal angular elastic deformation at the payload attach point is $\mathbf{a}_e = (\phi_e, \theta_e, \psi_e)^T = \Phi_\theta^T \delta \mathbf{q} = \mathbf{0}$, and the nominal gimbal angles are $\mathbf{a}_p = (\phi_p, \theta_p, \psi_p)^T = (\bar{\phi}_p, \bar{\theta}_p, \bar{\psi}_p)^T$. Furthermore, the modal coordinates used to describe the elastic motion are nominally zero, as are the forces and moments acting on the system.

The angular velocities of the various coordinate systems may be expressed in general as $\omega = \omega_o + \delta\omega$. However, the nominal angular rates are zero, and $\delta\omega = \delta\dot{\mathbf{a}}$, with $\mathbf{a} = (\phi, \theta, \psi)^T$, the Euler angle vector. The elastic deformation angular rate may be written in terms of the mode slopes and modal coordinates as follows: $\delta\dot{\mathbf{a}}_e = \Phi_\theta^T \delta\dot{\mathbf{q}}$.

- “Variable” = “Nominal” + “Perturbation”, $y = y_0 + \delta y$
- Nominal Values:
 - LFSP attitude, $\mathbf{a}_o = (\phi_o, \theta_o, \psi_o)^T = \mathbf{0}$
 - modal coordinates, $\mathbf{q} = \mathbf{0}$
 - gimbal angles, $\mathbf{a}_p = (\bar{\phi}_p, \bar{\theta}_p, \bar{\psi}_p)^T$

Linearized Model (cont.)

The linearized perturbation equations of motion are given below. The prime notation denotes the matrix cross-product form. $\bar{\mathbf{C}}_{ep}$ is the transformation from the $\hat{\mathbf{e}}$ -system to the $\hat{\mathbf{p}}$ -system for the nominal gimbal angles $\bar{\mathbf{a}}_p = (\bar{\phi}_p, \bar{\theta}_p, \bar{\psi}_p)^T$. Also, $\mathbf{I}_{po} = \bar{\mathbf{C}}_{ep}^T \mathbf{I}_p \bar{\mathbf{C}}_{ep}$ is the payload centroidal inertia matrix \mathbf{I}_p (expressed in the $\hat{\mathbf{p}}^*$ -system) transformed into the $\hat{\mathbf{o}}$ -system. \mathbf{I}_{po} premultiplies the payload inertial angular acceleration.

- LFSP translation:

$$m_o \delta \ddot{\mathbf{R}}_{no} = -\delta \mathbf{F}_p + \sum_{i=1}^{n_F} \delta \mathbf{F}_{oi} \quad (8)$$

- LFSP rotation:

$$\mathbf{I}_o \delta \ddot{\boldsymbol{\phi}}_o = -\mathbf{R}'_{op} \delta \mathbf{F}_p - \delta \mathbf{T}_p + \sum_{i=1}^{n_M} \delta \mathbf{M}_{oi} + \sum_{i=1}^{n_F} \mathbf{r}'_{oi} \delta \mathbf{F}_{oi} \quad (9)$$

- LFSP elastic motion:

$$\delta \ddot{\mathbf{q}} + \mathbf{D} \delta \dot{\mathbf{q}} + \mathbf{\Lambda} \delta \mathbf{q} = -\Phi_u \delta \mathbf{F}_p - \Phi_\theta \delta \mathbf{T}_p + \sum_{i=1}^{n_M} \Phi_{\theta oi} \delta \mathbf{M}_{oi} + \sum_{i=1}^{n_F} \Phi_{uoi} \delta \mathbf{F}_{oi} \quad (10)$$

- Payload CM translation:

$$m_p \{ \delta \ddot{\mathbf{R}}_{no} - [\mathbf{R}'_{op} + (\bar{\mathbf{C}}_{ep}^T \mathbf{r}_{p\bullet})'] \delta \ddot{\mathbf{a}}_o + [\Phi_u^T - (\bar{\mathbf{C}}_{ep}^T \mathbf{r}_{p\bullet})' \Phi_\theta^T] \delta \ddot{\mathbf{q}} - (\bar{\mathbf{C}}_{ep}^T \mathbf{r}_{p\bullet})' \bar{\mathbf{C}}_{ep}^T \delta \ddot{\mathbf{a}}_p \} = \delta \mathbf{F}_p \quad (11)$$

- Payload rotation:

$$\mathbf{I}_{po} (\delta \ddot{\mathbf{a}}_o + \Phi_\theta^T \delta \ddot{\mathbf{q}} + \bar{\mathbf{C}}_{ep}^T \delta \ddot{\mathbf{a}}_p) = \delta \mathbf{T}_p - (\bar{\mathbf{C}}_{ep}^T \mathbf{r}_{p\bullet})' \delta \mathbf{F}_p \quad (12)$$

Linearized Model (cont.)

The final equations of motion can be coupled by substituting for the constraint force $\delta \mathbf{F}_p$ given by equation (11) into the remaining equations (8, 9, 10, and 12), thus eliminating the expression for payload translation. The LFSP translational, rotational, and elastic equations, and the payload rotational equation may be expressed in the form of equation (13). Here \mathbf{A} is a (4×4) block matrix of overall dimension $(n + 9)$ square. Matrices \mathbf{B} and \mathbf{C} are essentially null except for n diagonal terms corresponding to the elastic motion. The vector $\delta \eta$ is $(n + 9) \times 1$, and is given by equation (14).

The expression for matrix \mathbf{A} is given by (15). Various terms comprising matrix \mathbf{A} are given by equation (16). The expressions for matrices \mathbf{B} and \mathbf{C} are given by equation (17). The matrices \mathbf{D} and \mathbf{A} were given previously. The mathematical structure of this generic model includes the internal couplings between payload rigid-body motion and LFSP rigid and elastic motion.

- Model form:

$$\mathbf{A} \delta \ddot{\eta} + \mathbf{B} \delta \dot{\eta} + \mathbf{C} \delta \eta = \mathbf{E} \delta \mathbf{u} \quad (13)$$

- Vector of system variables:

$$\delta \eta = \delta(X, Y, Z; \phi_o, \theta_o, \psi_o; q_1, \dots, q_n; \phi_p, \theta_p, \psi_p)^T \quad (14)$$

- Matrices \mathbf{A} , \mathbf{B} , and \mathbf{C} :

$$\mathbf{A} = \begin{bmatrix} (m_o + m_p) \mathbf{I}_{3 \times 3} & -m_p \bar{\mathbf{R}}' & m_p \bar{\Phi} & -m_p \bar{\mathbf{r}}' \bar{\mathbf{C}}_{ep}^T \\ m_p \mathbf{R}'_{op} & \mathbf{I}_o - m_p \mathbf{R}'_{op} \bar{\mathbf{R}}' & m_p \mathbf{R}'_{op} \bar{\Phi} & -m_p \mathbf{R}'_{op} \bar{\mathbf{r}}' \bar{\mathbf{C}}_{ep}^T \\ m_p \Phi_u & -m_p \Phi_u \bar{\mathbf{R}}' & \mathbf{I}_{n \times n} + m_p \Phi_u \bar{\Phi} & -m_p \Phi_u \bar{\mathbf{r}}' \bar{\mathbf{C}}_{ep}^T \\ m_p \bar{\mathbf{r}}' & \mathbf{I}_{po} - m_p \bar{\mathbf{r}}' \bar{\mathbf{R}}' & \mathbf{I}_{po} \Phi_\theta^T + m_p \bar{\mathbf{r}}' \bar{\Phi} & (\mathbf{I}_{po} - m_p \bar{\mathbf{r}}' \bar{\mathbf{r}}') \bar{\mathbf{C}}_{ep}^T \end{bmatrix} \quad (15)$$

where:

$$\bar{\mathbf{r}}' = (\bar{\mathbf{C}}_{ep}^T \mathbf{r}_{p\bullet})' \quad \bar{\mathbf{R}}' = (\mathbf{R}'_{op} + \bar{\mathbf{r}}') \quad \bar{\Phi} = (\Phi_u^T - \bar{\mathbf{r}}' \Phi_\theta^T) \quad (16)$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{0}_{6 \times 6} & & \\ & \mathbf{D}_{n \times n} & \\ & & \mathbf{0}_{3 \times 3} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{0}_{6 \times 6} & & \\ & \mathbf{A}_{n \times n} & \\ & & \mathbf{0}_{3 \times 3} \end{bmatrix} \quad (17)$$

Linearized Model (cont.)

Now consider the right-hand side of equation (13). The vector of control inputs is given by (18). Matrix \mathbf{E} is made up of n_F block-columns corresponding to forces $\delta \mathbf{F}_{oi}$, n_M block-columns for moments $\delta \mathbf{M}_{oi}$, and a single block-column multiplying the payload torque. The overall dimensions of matrix \mathbf{E} are $(n+9) \times (3n_F + 3n_M + 3)$, and \mathbf{E} as a partitioned matrix is given by equation (19). Each of the three partitioned matrices comprising \mathbf{E} are given in (20). It is clear that \mathbf{E}_1 corresponds to external force inputs to the LFSP, \mathbf{E}_2 to external moment inputs, and finally \mathbf{E}_3 corresponds to the payload gimbal torque.

- Vector of control inputs:

$$\delta \mathbf{u} = (\delta \mathbf{F}_{o1}^T, \dots, \delta \mathbf{F}_{on_F}^T; \delta \mathbf{M}_{o1}^T, \dots, \delta \mathbf{M}_{on_M}^T; \delta \mathbf{T}_p^T)^T \quad (18)$$

- Partitioned form of matrix \mathbf{E} :

$$\mathbf{E} = [\mathbf{E}_{1(n+9) \times (3n_F)}, \mathbf{E}_{2(n+9) \times (3n_M)}, \mathbf{E}_{3(n+9) \times 3}] \quad (19)$$

$$\mathbf{E}_1 = \begin{bmatrix} \mathbf{I}_{3 \times 3} & \dots & \mathbf{I}_{3 \times 3} \\ \mathbf{r}'_{o1} & \dots & \mathbf{r}'_{on_F} \\ \Phi_{uo1} & \dots & \Phi_{uon_F} \\ \mathbf{0}_{3 \times 3} & \dots & \mathbf{0}_{3 \times 3} \end{bmatrix}, \mathbf{E}_2 = \begin{bmatrix} \mathbf{0}_{3 \times 3} & \dots & \mathbf{0}_{3 \times 3} \\ \mathbf{I}_{3 \times 3} & \dots & \mathbf{I}_{3 \times 3} \\ \Phi_{\theta o1} & \dots & \Phi_{\theta on_M} \\ \mathbf{0}_{3 \times 3} & \dots & \mathbf{0}_{3 \times 3} \end{bmatrix}, \mathbf{E}_3 = \begin{bmatrix} \mathbf{0}_{3 \times 3} \\ -\mathbf{I}_{3 \times 3} \\ -\Phi_\theta \\ \mathbf{I}_{3 \times 3} \end{bmatrix} \quad (20)$$

Decentralized Control Law

The matrix equation (13) may be put into state variable form by allowing $\mathbf{x}_1 = \delta\eta$, as given in (14), and $\mathbf{x}_2 = \delta\dot{\eta}$. Thus one obtains the expressions in (21), which may in turn be written as equation (22). The matrices $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ are given by equation (23), with each sub-block of $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ dimensioned $(9 + n)$ square.

The approach considered herein is a decentralized control strategy using simple proportional plus rate feedback with collocated actuators and sensors for both payload pointing and LFSP attitude hold. Only applied moments are considered, with LFSP attitude control effected by moment input $\delta\mathbf{M}_o$, and payload pointing accomplished by an applied gimbal torque $\delta\mathbf{T}_p$. The sensed LFSP attitude and attitude rate at the CMG location are given by (24). The LFSP attitude is the sum of the central body rigid angles, and the elastic rotations at the CMG location. The sensed attitude and attitude rate of the payload are given by (25). The payload attitude is the sum of the central body rigid angles, the elastic rotation angles at the payload attach point, and the gimbal angles.

• State-space Form

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= \mathbf{A}^{-1}[\mathbf{E}\mathbf{u} - \mathbf{B}\mathbf{x}_2 - \mathbf{C}\mathbf{x}_1]\end{aligned}\tag{21}$$

$$\dot{\mathbf{x}} = \bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{B}}\mathbf{u}\tag{22}$$

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{A}^{-1}\mathbf{C} & -\mathbf{A}^{-1}\mathbf{B} \end{bmatrix} \quad \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{A}^{-1}\mathbf{E} \end{bmatrix}\tag{23}$$

• Sensor Outputs

- LFSP attitude and attitude rate:
- Payload attitude and attitude rate:

$$\begin{aligned}\delta\mathbf{a}_{os} &= \delta\mathbf{a}_o + \Phi_{\theta o}^T \delta\mathbf{q} \\ \delta\dot{\mathbf{a}}_{os} &= \delta\dot{\mathbf{a}}_o + \Phi_{\theta o}^T \delta\dot{\mathbf{q}}\end{aligned}\tag{24}$$

$$\begin{aligned}\delta\mathbf{a}_{ps} &= \delta\mathbf{a}_o + \Phi_{\theta}^T \delta\mathbf{q} + \bar{\mathbf{C}}_{ep}^T \delta\mathbf{a}_p \\ \delta\dot{\mathbf{a}}_{ps} &= \delta\dot{\mathbf{a}}_o + \Phi_{\theta}^T \delta\dot{\mathbf{q}} + \bar{\mathbf{C}}_{ep}^T \delta\dot{\mathbf{a}}_p\end{aligned}\tag{25}$$

Decentralized Control Law (cont.)

The decentralized control laws for LFSP attitude control and payload pointing are given by equations (26) and (27), respectively. Consider the LFSP attitude control system. If the gain matrices $\mathbf{K}_{\mathbf{a}_o}$ and $\mathbf{K}_{\dot{\mathbf{a}}_o}$ are chosen as in (28), the resulting closed-loop system (assuming no coupling) would have a damping ratio of ρ_o , and a bandwidth of Ω_o rad/sec. Similar expressions for the payload gimbal controller may be written by replacing \mathbf{A}_{22} with \mathbf{A}_{44} as in equation (29), giving a closed-loop system with damping ratio ρ_p and bandwidth Ω_p .

• Control Law

– LFSP attitude:

$$\delta \mathbf{M}_o = -[\mathbf{K}_{\mathbf{a}_o} \delta \mathbf{a}_{os} + \mathbf{K}_{\dot{\mathbf{a}}_o} \delta \dot{\mathbf{a}}_{os}] \quad (26)$$

– Payload Gimbal Torquer:

$$\delta \mathbf{T}_p = -[\mathbf{K}_{\mathbf{a}_p} \delta \mathbf{a}_{ps} + \mathbf{K}_{\dot{\mathbf{a}}_p} \delta \dot{\mathbf{a}}_{ps}] \quad (27)$$

– LFSP Gain Matrices:

$$\begin{aligned} \mathbf{K}_{\mathbf{a}_o} &= -\mathbf{A}_{22} \text{diag}(\Omega_o^2) \\ \mathbf{K}_{\dot{\mathbf{a}}_o} &= -\mathbf{A}_{22} \text{diag}(2\rho_o \Omega_o) \end{aligned} \quad (28)$$

– Payload Gain Matrices:

$$\begin{aligned} \mathbf{K}_{\mathbf{a}_p} &= -\mathbf{A}_{44} \text{diag}(\Omega_p^2) \\ \mathbf{K}_{\dot{\mathbf{a}}_p} &= -\mathbf{A}_{44} \text{diag}(2\rho_p \Omega_p) \end{aligned} \quad (29)$$

Extension to Multi-Payload Case

The equations of motion for multi-payload configurations may be written in the form of equation (30). The vector $\delta\eta$ is dimensioned $(6 + n + 3n_p) \times 1$, and is given by (31). The vector of force and moment inputs $\delta\mathbf{u}$ is dimensioned $(3n_F + 3n_M + 3n_p) \times 1$, and is written as in (32), where $\delta\mathbf{F}_{oi}$, $\delta\mathbf{M}_{oi}$, and $\delta\mathbf{T}_i$ are external LFSP forces and moments, and payload gimbal torques, respectively. The coefficient matrices appearing in equation (30) are given in Appendix A.

- matrix form of eqns.:

$$\mathbf{A}\delta\ddot{\eta} + \mathbf{B}\delta\dot{\eta} + \mathbf{C}\delta\eta = \mathbf{E}\delta\mathbf{u} \quad (30)$$

- vector of system variables:

$$\delta\eta = \delta(X, Y, Z; \phi_o, \theta_o, \psi_o; q_1, \dots, q_n; \phi_1, \theta_1, \psi_1; \dots; \phi_{n_p}, \theta_{n_p}, \psi_{n_p})^T \quad (31)$$

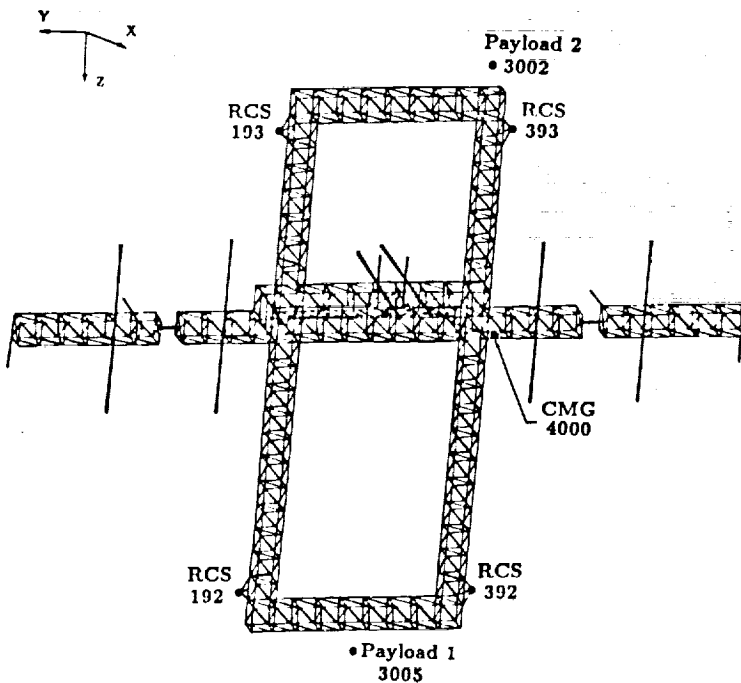
- vector of control inputs:

$$\delta\mathbf{u} = \delta(\mathbf{F}_{o1}^T, \dots, \mathbf{F}_{on_F}^T; \mathbf{M}_{o1}^T, \dots, \mathbf{M}_{on_M}^T; \mathbf{T}_1^T, \dots, \mathbf{T}_{n_p}^T)^T \quad (32)$$

Simulation Results for an Example Problem

In order to demonstrate the modeling and control methods developed, the dual-keel space station ISS04 reference configuration with individually pointed payloads was considered. Mass, inertia, and other parameters for the station are summarized in the table below. Many different simulation scenarios were considered in exercising the space station model, however only a two-payload study comparing the effects of end-mounting versus CM-mounting will be presented herein. Attitude control moment input to the station was provided by a three-axis CMG near the station CM, while three-axis gimbal torquers acted at each of the two payload attachment points. In this study, the "worst-case" station rigid-body initial conditions were chosen based on the most severe anticipated maneuver, re-boosting to higher orbit. Initial modal coordinates were obtained from the steady-state elastic deformation resulting from the firing of RCS thrusters. The decentralized controller must drive the station rigid-body angles and payload pointing errors to zero in the presence of elastic motion.

ISS04 Configuration



c.m. location: (ft)	$X = -9.1667$ $Y = -5.4167$ $Z = -21.333$
mass: (slug)	$m_o = 17757.76$
inertias w.r.t. CM: (slug-ft ²)	$I_{xx} = 2.15670 \times 10^8$ $I_{yy} = 2.15450 \times 10^7$ $I_{zz} = 3.27812 \times 10^5$ $I_{xy} = 1.20342 \times 10^8$ $I_{yz} = 1.06323 \times 10^6$ $I_{xz} = 1.19695 \times 10^8$
RCS nodes: CMG node:	192, 193, 392, 392 4000

Simulation Results (cont.)

The 12 vibrational modes that were actually used in the simulation studies, and their corresponding modal frequencies, are tabulated below. These 12 modes were chosen because they represent major truss-structure bending and torsional deformations, as opposed to mere appendage modes. The elastic modes were assumed to have a damping ratio of 0.005. Payload data are also summarized below. Given are the payload attachment points, and the masses and inertias. The first rigid payload was mounted on the lower keel, and the second payload on the upper keel. For the control systems, the desired closed-loop damping ratios were assumed to be 0.707, and the desired bandwidths were 10.0 rad/sec and 0.1 rad/sec for the two payloads and the space station, respectively.

Model Data

mode #	frequency (rad/sec)
7	1.37236
8	1.39753
9	1.50451
20	2.01425
21	2.24925
22	2.68481
36	4.09370
37	4.74973
38	4.87077
39	5.86502
40	6.51971
41	7.28311

Modes Included in Composite System Model

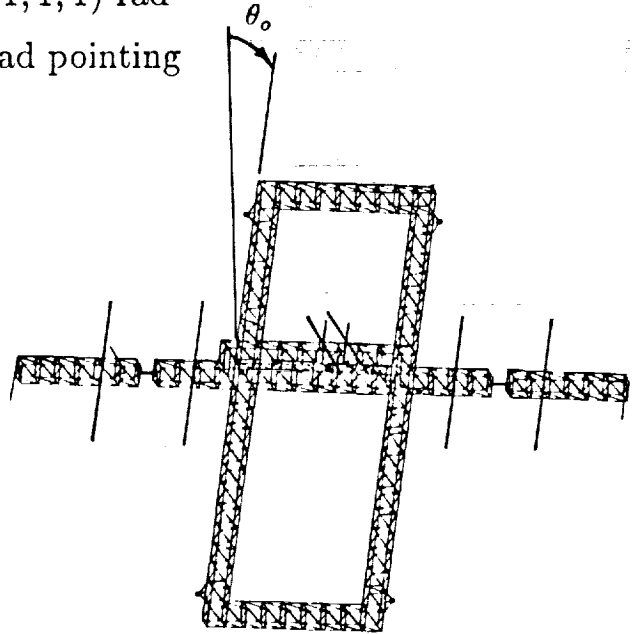
Payload Data

	payload 1	payload 2
ISS04 gimbal node	3005	3002
position vectors from CM (ft)	$x = 8.3333$ $y = 5.4167$ $z = 236.223$	$x = 9.1667$ $y = -60.202$ $z = -142.708$
payload mass (slug)	100.0	100.0
payload inertias (slug-ft ²)		
I_{xx}	1250.0	1250.0
I_{xy}	0.0	0.0
I_{xz}	0.0	0.0
I_{yy}	1250.0	1250.0
I_{yz}	0.0	0.0
I_{zz}	1500.0	1500.0

Simulation Results (cont.)

The steady-state (limit cycle) station rigid-body pitch and pitch rate encountered in a reboost maneuver simulation study were 0.02187 rad (1.25 deg) and 0.0008727 rad/sec (0.05 deg/sec), respectively. The present simulation used these values as initial angular displacements and velocities about each of the station axes. The initial modal coordinates were calculated by assuming the space station had reached steady-state conditions after the firing of the four RCS thrusters in the x -direction at 75 lb_f each. This assumed maneuver serves only to provide a reasonable initial deformed state. There are no thruster inputs after the start of the simulation. The payload gimbal Euler angles and the remaining states are initially zero. The desired gimbal angles for both payloads are $\bar{a}_{1,2} = 0.1(1,1,1)$ rad. It was desired to return the station to its nominal orientation, i.e. the Euler angles $(\phi_o, \theta_o, \psi_o) = 0$, while driving the payload pointing error to zero, in the presence of nonzero initial conditions and any induced elastic deformation. Two cases were compared as a result of studying the equations of motion: an end-mounted case with nonzero distance between the gimbals and the CM, $\mathbf{r}_{p*} = (0, 0, 6)^T$ ft, and a CM-mounted case where $\mathbf{r}_{p*} = (0, 0, 0)^T$.

- Initial Conditions
 - each axis: angle = 1.25 deg, angular rate = 0.05 deg/sec
 - initial elastic deformation due to thruster firing (steady-state)
 - initial gimbal angles are all zero
- desired gimbal angles: $\mathbf{a}_{p1} = \mathbf{a}_{p2} = 0.1(1,1,1)$ rad
- station stabilization, simultaneous payload pointing
- two cases:
 - end-mounted payload
 - CM-mounted payload

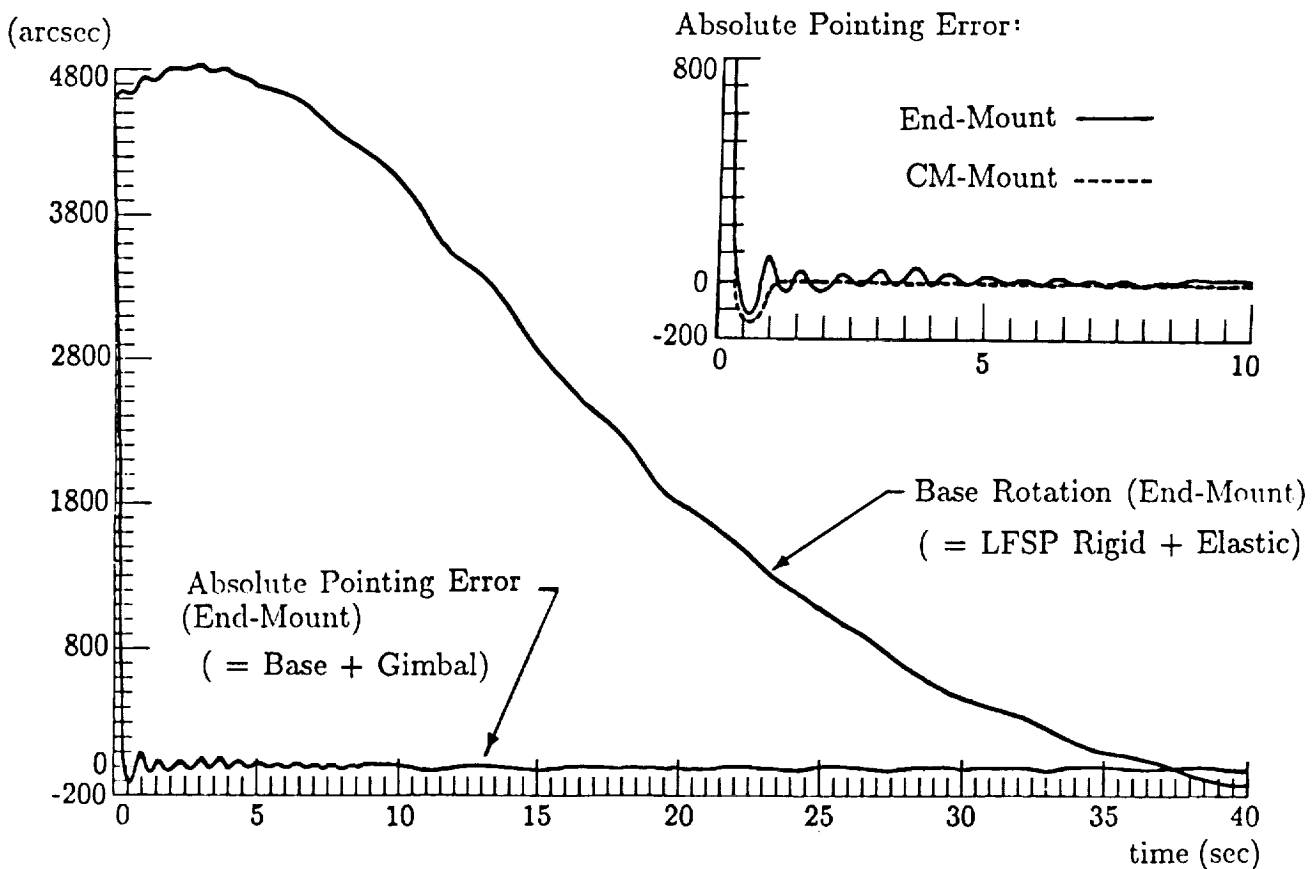


Simulation Results (cont.)

The results given below are for Payload 2 on the upper keel, about the pitch axis (the x and z motions are analogous, as are the results for the payload on the lower keel). The figure shows the "base" motion, that is, the sum of the space station rigid-body orientation angles and the elastic rotation at the payload attachment point, for an end-mounted payload. This motion must be compensated for by the payload gimbal torquer so that the payload remains on target. The large rigid-body motion about the y -axis masks the much smaller elastic motion. The absolute payload pointing error for the end-mounted case is also shown below. The absolute pointing error is the sum of the base motion and the payload gimbal angles, and should be driven to zero by the payload controller.

The inset figure compares the absolute pointing error for the CM- and end-mounted cases. Note that the absolute error for the end-mounted case is significantly more oscillatory than the CM-mounted case. The end-mounted case has a maximum overshoot of -136.49 arcsec. This response remains oscillatory with a magnitude on the order of 10 arcsec. The CM-mounted case yields the desired error response (i.e., perfect second-order system response) about the y -axis (and x and z axes as well) because the payload rotational motion is completely uncoupled from the LFSP motion. Here the overshoot is -194.22 arcsec, with the absolute error dropping below 0.001 arcsec in 2.32 sec.

- Results for Payload 2, angles about pitch axis (i.e.: " $\delta\theta$ ")



Alternate Dissipative Controller

The decentralized controller described previously was found to give satisfactory performance for the space station example. However, coupling between rigid and elastic motion was not severe, as the masses and inertias of the payloads were small compared to the station values, and the station represents a nearly rigid spacecraft. Decentralized control does not guarantee stability for the general case of a highly flexible LFSP with large inertial payload/LFSP coupling, as control system interactions can be destabilizing. An alternate method is to utilize a dissipative controller which requires that the matrix "A" in the second order vector equation (33) be symmetric. Development of the symmetric form of the equations of motion may be found in Appendix B. By redefining the system variables " $\delta\eta$ " as in (35), it can be shown that the resulting "A" matrix is always positive definite. Using the sensor outputs as in (36) and the control law as in (37), it can be proven that the system is asymptotically stable (in the sense that all rotational and elastic motion tends to zero as time goes to infinity). The stability is guaranteed regardless of errors in the system parameters or ignored (unmodeled) elastic motion. Research is presently in progress on this type of controller.

• Symmetric Form of Equations

$$A\delta\ddot{\eta} + B\delta\dot{\eta} + C\delta\eta = E\delta u \quad (33)$$

where:

$$\begin{aligned} A^T &= A > 0 \\ B^T &= B \geq 0 \\ C^T &= C \geq 0 \end{aligned} \quad E = \left[\begin{array}{c|c} \begin{matrix} 0_{3 \times 3} \\ I_{3 \times 3} \\ \Phi_{\theta o(n \times 3)} \\ 0_{3 \times 3} \\ \vdots \\ 0_{3 \times 3} \end{matrix} & \begin{matrix} 0_{[(6+n) \times 3n_p]} \\ \\ \\ I_{(3n_p \times 3n_p)} \end{matrix} \end{array} \right] \quad (34)$$

$$\delta\eta = \left(\delta R_{no}^T, \delta a_o^T, \delta q^T, \delta a_{1o}^T, \dots, \delta a_{npo}^T \right)^T \quad (35)$$

$$y = E^T \eta = \left[\begin{array}{l} \text{station attitude (rigid + flexible), at CMG location} \\ \text{transformed gimbal angles, payload 1} \\ \vdots \\ \text{transformed gimbal angles, payload } n_p \end{array} \right] \quad (36)$$

• Control Law

$$\delta u = -G_p \delta y - G_r \delta \dot{y}; \quad G_p^T = G_p > 0, \quad G_r^T = G_r > 0 \quad (37)$$

• Closed-loop system asymptotically stable, i.e.:

$$(\delta a_o, \delta q, \delta a_{1o}, \delta a_{2o}, \dots, \delta a_{npo}) \rightarrow 0 \text{ as } t \rightarrow \infty \quad (38)$$

• Research presently in progress

Concluding Remarks

The problem of dynamic modeling and control of composite systems consisting of a central flexible space platform and articulated rigid payloads was considered. Generic linearized equations of motion were derived for a single rigid payload mounted on such a platform using a three-degree-of-freedom gimbaling mechanism. This generic large flexible space platform (LFSP) model was then extended to the case of multiple independently pointed payloads. A simple decentralized control law was proposed for platform attitude control and payload pointing. Simulation results were obtained for an example problem with a single payload, and also with two payloads, attached to the space station. The results demonstrate the effects of dynamic couplings in the composite system, and also indicate that the control law used provides satisfactory payload pointing and platform stabilization for the example problem.

An alternate centralized dissipative control law, which uses a symmetric form of the equations of motion, was also proposed. This control law guarantees stability regardless of modeling errors and unmodeled modes. Further research is in progress on that topic. The model obtained herein is mathematically tractable, and yet has an accurate structure that includes all internal dynamic couplings. This model offers a suitable tool for analytical and numerical investigation of the dynamics and control of an important class of missions arising in the Control-Structure Interaction (CSI) program.

- Developed nonlinear math models for LFSP/ articulated payload system
- Extended to multi-payload systems
- Obtained linearized equations
- Simulation results for an example problem
 - indicates satisfactory performance for decentralized control law
- Proposed a centralized dissipative control law
 - provides robust stability
 - development in progress
- Further investigation:
 - extension to flexible articulated appendages
 - automated “FEM-type” model assembly
 - incorporation in CSI optimization problem

Appendix A: Equations for Multi-Payload Case

The equations for an LFSP supporting multiple payloads will be derived from the linearized equations (8) through (12). Here a large flexible space platform described in part by n known elastic modes, is acted on by n_F external forces and n_M moments exclusive of the torques required to point the n_p payloads.

The translational expression is:

$$m_o \delta \ddot{\mathbf{R}}_{no} = \sum_{i=1}^{n_F} \delta \mathbf{F}_{oi} - \sum_{i=1}^{n_p} \delta \mathbf{F}_i \quad (A1)$$

The expression for the rigid-body rotation of the LFSP becomes:

$$\mathbf{I}_o \delta \ddot{\mathbf{a}}_o = \sum_{i=1}^{n_M} \delta \mathbf{M}_{oi} + \sum_{i=1}^{n_F} \mathbf{r}'_{oi} \delta \mathbf{F}_{oi} - \sum_{i=1}^{n_p} \mathbf{R}'_{oi} \delta \mathbf{F}_i - \sum_{i=1}^{n_p} \delta \mathbf{T}_i \quad (A2)$$

Again the "prime" notation with \mathbf{R}'_{oi} and \mathbf{r}'_{oi} indicates the (3×3) matrix cross-product form of the vectors $\mathbf{R}_{oi} = x_i \hat{\mathbf{o}}_1 + y_i \hat{\mathbf{o}}_2 + z_i \hat{\mathbf{o}}_3$ and $\mathbf{r}_{oi} = x_{oi} \hat{\mathbf{o}}_1 + y_{oi} \hat{\mathbf{o}}_2 + z_{oi} \hat{\mathbf{o}}_3$. The linearized equation for the elastic motion of the platform is:

$$\delta \ddot{\mathbf{q}} + \mathbf{D} \delta \dot{\mathbf{q}} + \mathbf{A} \delta \mathbf{q} = \sum_{i=1}^{n_M} \Phi_{\theta oi} \delta \mathbf{M}_{oi} + \sum_{i=1}^{n_F} \Phi_{u oi} \delta \mathbf{F}_{oi} - \sum_{i=1}^{n_p} \Phi_{ui} \delta \mathbf{F}_i - \sum_{i=1}^{n_p} \Phi_{\theta i} \delta \mathbf{T}_i \quad (A3)$$

Next are the rigid payload equations. There will be a translational and a rotational equation for each of the n_p payloads. The CM of the i^{th} payload ("point i^* ") from the inertial reference for the deformed spacecraft is given by:

$$\mathbf{R}_{ni^*} = \mathbf{R}_{no} + \mathbf{R}_{oi} + \Phi_{ui}^T \mathbf{q} + \mathbf{r}_{i^*} \quad (A4)$$

Differentiating this equation twice results in the inertial acceleration of the i^{th} payload CM in the form of equation (7). Noting that the only external force acting on the payload is the constraint force, and linearizing gives:

$$m_i \{ \delta \ddot{\mathbf{R}}_{no} - [\mathbf{R}'_{oi} + (\bar{\mathbf{C}}_{ei}^T \mathbf{r}_{i^*})'] \delta \ddot{\mathbf{a}}_o + [\Phi_{ui}^T - (\bar{\mathbf{C}}_{ei}^T \mathbf{r}_{i^*})' \Phi_{\theta i}^T] \delta \ddot{\mathbf{q}} - (\bar{\mathbf{C}}_{ei}^T \mathbf{r}_{i^*})' \bar{\mathbf{C}}_{ei}^T \delta \ddot{\mathbf{a}}_i \} = \delta \mathbf{F}_i \quad (A5)$$

$\bar{\mathbf{C}}_{ei}$ is the nominal transformation between the elastic and some i^{th} payload axes. The payload rotational equation of motion becomes:

$$\mathbf{I}_{io} (\delta \ddot{\mathbf{a}}_o + \Phi_{\theta i}^T \delta \ddot{\mathbf{q}} + \bar{\mathbf{C}}_{ei}^T \delta \ddot{\mathbf{a}}_i) = \delta \mathbf{T}_i - (\bar{\mathbf{C}}_{ei}^T \mathbf{r}_{i^*})' \delta \mathbf{F}_i \quad (A6)$$

Here $\mathbf{I}_{i\mathbf{o}} = \bar{\mathbf{C}}_{\mathbf{e}\mathbf{i}}^T \mathbf{I}_i \bar{\mathbf{C}}_{\mathbf{e}\mathbf{i}}$ is the i^{th} payload inertia matrix transformed into the $\hat{\mathbf{o}}$ -system. This inertia matrix multiplies the absolute inertial angular acceleration of the i^{th} payload.

Using summations over the n_p payloads, equation (A5) can be used to remove $\sum \delta \mathbf{F}_i$ terms from the LFSP equations (A1, A2, and A3). Furthermore, $\delta \mathbf{F}_i$ may be removed from the i^{th} payload rotational equation (A6), yielding n_p additional equations. This system of equations may then be assembled in the following matrix form:

$$\mathbf{A}\delta\ddot{\boldsymbol{\eta}} + \mathbf{B}\delta\dot{\boldsymbol{\eta}} + \mathbf{C}\delta\boldsymbol{\eta} = \mathbf{E}\delta\mathbf{u} \quad (\text{A7})$$

The vector $\delta\boldsymbol{\eta}$ is dimensioned $(6 + n + 3n_p) \times 1$, and is given by:

$$\delta\boldsymbol{\eta} = \delta(X, Y, Z; \phi_o, \theta_o, \psi_o; q_1, \dots, q_n; \phi_1, \theta_1, \psi_1; \dots; \phi_{n_p}, \theta_{n_p}, \psi_{n_p})^T \quad (\text{A8})$$

The vector of force and moment inputs $\delta\mathbf{u}$ is dimensioned $(3n_F + 3n_M + 3n_p) \times 1$, and is written as follows:

$$\delta\mathbf{u} = \delta(\mathbf{F}_{\mathbf{o}\mathbf{1}}^T, \dots, \mathbf{F}_{\mathbf{o}\mathbf{n}_F}^T; \mathbf{M}_{\mathbf{o}\mathbf{1}}^T, \dots, \mathbf{M}_{\mathbf{o}\mathbf{n}_M}^T; \mathbf{T}_1^T, \dots, \mathbf{T}_{n_p}^T)^T \quad (\text{A9})$$

where $\delta\mathbf{F}_{\mathbf{o}\mathbf{i}}$, $\delta\mathbf{M}_{\mathbf{o}\mathbf{i}}$, and $\delta\mathbf{T}_i$ are external LFSP forces and moments, and payload gimbal torques, respectively.

The coefficient matrices \mathbf{A} , \mathbf{B} , and \mathbf{C} , dimensioned $(6 + n + 3n_p)$ square, and \mathbf{E} , dimensioned $(6 + n + 3n_p) \times (3n_F + 3n_M + 3n_p)$, are given below. Matrix \mathbf{A} is essentially a “mass/inertia” matrix given by:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1[(6+n) \times (6+n)] & \mathbf{A}_2[(6+n) \times 3n_p] \\ \mathbf{A}_3[3n_p \times (6+n)] & \mathbf{A}_4[3n_p \times 3n_p] \end{bmatrix} \quad (\text{A10})$$

The blocks comprising matrix \mathbf{A} are:

$$\mathbf{A}_1 = \begin{bmatrix} \left(m_o + \sum_{i=1}^{n_p} m_i \right) \mathbf{I}_{3 \times 3} & - \sum_{i=1}^{n_p} m_i \bar{\mathbf{R}}'_i & \sum_{i=1}^{n_p} m_i \bar{\boldsymbol{\Phi}}_i \\ \sum_{i=1}^{n_p} m_i \mathbf{R}'_{\mathbf{o}\mathbf{i}} & \mathbf{I}_o - \sum_{i=1}^{n_p} m_i \mathbf{R}'_{\mathbf{o}\mathbf{i}} \bar{\mathbf{R}}'_i & \sum_{i=1}^{n_p} m_i \mathbf{R}'_{\mathbf{o}\mathbf{i}} \bar{\boldsymbol{\Phi}}_i \\ \sum_{i=1}^{n_p} m_i \boldsymbol{\Phi}_i & - \sum_{i=1}^{n_p} m_i \boldsymbol{\Phi}_i \bar{\mathbf{R}}'_i & \mathbf{I}_{n \times n} + \sum_{i=1}^{n_p} m_i \boldsymbol{\Phi}_i \bar{\boldsymbol{\Phi}}_i \end{bmatrix} \quad (\text{A11a})$$

$$\mathbf{A}_2 = \begin{bmatrix} -m_1 \bar{\mathbf{r}}'_1 \mathbf{C}_{\mathbf{e}\mathbf{1}}^T & \dots & -m_{n_p} \bar{\mathbf{r}}'_{n_p} \mathbf{C}_{\mathbf{e}\mathbf{n}_p}^T \\ m_1 \mathbf{R}'_{\mathbf{o}\mathbf{1}} \bar{\mathbf{r}}'_1 \mathbf{C}_{\mathbf{e}\mathbf{1}}^T & \dots & m_{n_p} \mathbf{R}'_{\mathbf{o}\mathbf{n}_p} \bar{\mathbf{r}}'_{n_p} \mathbf{C}_{\mathbf{e}\mathbf{n}_p}^T \\ -m_1 \boldsymbol{\Phi}_1 \bar{\mathbf{r}}'_1 \mathbf{C}_{\mathbf{e}\mathbf{1}}^T & \dots & -m_{n_p} \boldsymbol{\Phi}_{n_p} \bar{\mathbf{r}}'_{n_p} \mathbf{C}_{\mathbf{e}\mathbf{n}_p}^T \end{bmatrix} \quad (\text{A11b})$$

$$\mathbf{A}_3 = \begin{bmatrix} \bar{\mathbf{r}}'_1 m_1 & \mathbf{I}_1 - m_1 \bar{\mathbf{r}}'_1 \bar{\mathbf{R}}'_1 & \mathbf{I}_1 \Phi_{\theta 1}^T + m_1 \bar{\mathbf{r}}'_1 \bar{\Phi}_1 \\ \vdots & \vdots & \vdots \\ \bar{\mathbf{r}}'_{n_p} m_{n_p} & \mathbf{I}_{n_p} - m_{n_p} \bar{\mathbf{r}}'_{n_p} \bar{\mathbf{R}}'_{n_p} & \mathbf{I}_{n_p} \Phi_{\theta n_p}^T + m_{n_p} \bar{\mathbf{r}}'_{n_p} \bar{\Phi}_{n_p} \end{bmatrix} \quad (\text{A11c})$$

$$\mathbf{A}_4 = \begin{bmatrix} (\mathbf{I}_1 - m_1 \bar{\mathbf{r}}'_1 \bar{\mathbf{r}}'_1) \mathbf{C}_{e1}^T & \dots & \mathbf{0}_{3 \times 3} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{3 \times 3} & \dots & (\mathbf{I}_{n_p} - m_{n_p} \bar{\mathbf{r}}'_{n_p} \bar{\mathbf{r}}'_{n_p}) \mathbf{C}_{en_p}^T \end{bmatrix} \quad (\text{A11d})$$

where:

$$\bar{\mathbf{r}}'_i = (\mathbf{C}_{ei}^T \mathbf{r}_i)' \quad \bar{\mathbf{R}}'_i = (\mathbf{R}'_{oi} + \bar{\mathbf{r}}'_i) \quad \bar{\Phi}_i = (\Phi_{ui}^T - \bar{\mathbf{r}}'_i \Phi_{\theta i}^T) \quad (\text{A12})$$

Matrices \mathbf{B} and \mathbf{C} are for the most part null except for diagonal blocks multiplying the n modal coordinates. \mathbf{B} and \mathbf{C} are then modal damping and modal stiffness matrices written as:

$$\begin{aligned} \mathbf{B} &= \text{diag}[\mathbf{0}_{(6 \times 6)}, \mathbf{D}_{(n \times n)}, \mathbf{0}_{(3n_p \times 3n_p)}] \\ \mathbf{C} &= \text{diag}[\mathbf{0}_{(6 \times 6)}, \mathbf{\Lambda}_{(n \times n)}, \mathbf{0}_{(3n_p \times 3n_p)}] \end{aligned} \quad (\text{A13})$$

where:

$$\begin{aligned} \mathbf{D} &= [\text{diag}(2\zeta_i \omega_i)]_{(n \times n)} \\ \mathbf{\Lambda} &= [\text{diag}(\omega_i^2)]_{(n \times n)} \end{aligned} \quad (\text{A14})$$

The coefficient matrix \mathbf{E} is written as:

$$\mathbf{E} = \begin{bmatrix} \mathbf{E}_1[(6+n) \times 3n_F] & \mathbf{E}_2[(6+n) \times 3n_M] & \mathbf{E}_3[(6+n) \times 3n_p] \\ \mathbf{0}_{(3n_p \times 3n_F)} & \mathbf{0}_{(3n_p \times 3n_M)} & \mathbf{I}_{(3n_p \times 3n_p)} \end{bmatrix} \quad (\text{A15})$$

where the first row of blocks in \mathbf{E} are given by:

$$\mathbf{E}_1 = \begin{bmatrix} \mathbf{I}_{3 \times 3} & \dots & \mathbf{I}_{3 \times 3} \\ \mathbf{r}'_{o1} & \dots & \mathbf{r}'_{on_F} \\ \Phi_{uo1} & \dots & \Phi_{uon_F} \end{bmatrix}, \quad \mathbf{E}_2 = \begin{bmatrix} \mathbf{0}_{3 \times 3} & \dots & \mathbf{0}_{3 \times 3} \\ \mathbf{I}_{3 \times 3} & \dots & \mathbf{I}_{3 \times 3} \\ \Phi_{\theta o1} & \dots & \Phi_{\theta on_M} \end{bmatrix}, \quad \mathbf{E}_3 = \begin{bmatrix} \mathbf{0}_{3 \times 3} & \dots & \mathbf{0}_{3 \times 3} \\ -\mathbf{I}_{3 \times 3} & \dots & -\mathbf{I}_{3 \times 3} \\ -\Phi_{\theta 1} & \dots & -\Phi_{\theta n_p} \end{bmatrix} \quad (\text{A16})$$

Appendix B: Symmetric-Form Equations for Single and Multi-Payload Cases

A symmetric form of the composite system equations of motion will now be derived from the linearized asymmetric form. The linearized LFSP translation, rotation, and elastic equations of motion, as well as the payload rotational equations follow:

$$(m_o + m_p)\mathbf{I}_{3 \times 3}\delta\ddot{\mathbf{R}}_{\text{no}} - m_p\bar{\mathbf{R}}'\delta\ddot{\mathbf{a}}_o + m_p\bar{\Phi}\delta\ddot{\mathbf{q}} - m_p\bar{\mathbf{r}}'\bar{\mathbf{C}}_{\text{ep}}^T\delta\ddot{\mathbf{a}}_p = \sum_{i=1}^{n_F}\delta\mathbf{F}_{oi} \quad (B1)$$

$$\begin{aligned} m_p\mathbf{R}'_{\text{op}}\delta\ddot{\mathbf{R}}_{\text{no}} + [\mathbf{I}_o - m_p\mathbf{R}'_{\text{op}}\bar{\mathbf{R}}']\delta\ddot{\mathbf{a}}_o + m_p\mathbf{R}'_{\text{op}}\bar{\Phi}\delta\ddot{\mathbf{q}} - m_p\mathbf{R}'_{\text{op}}\bar{\mathbf{r}}'\bar{\mathbf{C}}_{\text{ep}}^T\delta\ddot{\mathbf{a}}_p \\ = \sum_{i=1}^{n_F}\mathbf{r}'_{oi}\delta\mathbf{F}_{oi} + \sum_{i=1}^{n_M}\delta\mathbf{M}_{oi} - \delta\mathbf{T}_p \end{aligned} \quad (B2)$$

$$\begin{aligned} m_p\Phi_u\delta\ddot{\mathbf{R}}_{\text{no}} - m_p\Phi_u\bar{\mathbf{R}}'\delta\ddot{\mathbf{a}}_o + [\mathbf{I}_{n \times n} + m_p\Phi_u\bar{\Phi}]\delta\ddot{\mathbf{q}} - m_p\Phi_u\bar{\mathbf{r}}'\bar{\mathbf{C}}_{\text{ep}}^T\delta\ddot{\mathbf{a}}_p \\ = \sum_{i=1}^{n_F}\Phi_{uoi}\delta\mathbf{F}_{oi} + \sum_{i=1}^{n_M}\Phi_{\theta oi}\delta\mathbf{M}_{oi} - \Phi_\theta\delta\mathbf{T}_p \end{aligned} \quad (B3)$$

$$\begin{aligned} m_p\bar{\mathbf{r}}'\delta\ddot{\mathbf{R}}_{\text{no}} + [\mathbf{I}_{p_o} - m_p\bar{\mathbf{r}}'\bar{\mathbf{R}}']\delta\ddot{\mathbf{a}}_o + [\mathbf{I}_{p_o}\Phi_\theta^T + m_p\bar{\mathbf{r}}'\bar{\Phi}]\delta\ddot{\mathbf{q}} \\ + (\mathbf{I}_{p_o} - m_p\bar{\mathbf{r}}'\bar{\mathbf{r}}')\bar{\mathbf{C}}_{\text{ep}}^T\delta\ddot{\mathbf{a}}_p = \delta\mathbf{T}_p \end{aligned} \quad (B4)$$

where:

$$\bar{\mathbf{r}}' = (\bar{\mathbf{C}}_{\text{ep}}^T\mathbf{r}_{p*})' \quad \bar{\mathbf{R}}' = (\mathbf{R}'_{\text{op}} + \bar{\mathbf{r}}') \quad \bar{\Phi} = (\Phi_u^T - \bar{\mathbf{r}}'\Phi_\theta^T) \quad (B5)$$

A symmetric form of the equations of motion may now be obtained by removing $\delta\mathbf{T}_p$ from equations (B2) and (B3) using (B4), yielding:

$$\begin{aligned} m_p\bar{\mathbf{R}}'\delta\ddot{\mathbf{R}}_{\text{no}} + (\mathbf{I}_o + \mathbf{I}_{p_o} - m_p\bar{\mathbf{R}}'\bar{\mathbf{R}}')\delta\ddot{\mathbf{a}}_o + (\mathbf{I}_{p_o}\Phi_\theta^T + m_p\bar{\mathbf{R}}'\bar{\Phi})\delta\ddot{\mathbf{q}} \\ + (\mathbf{I}_{p_o} - m_p\bar{\mathbf{R}}'\bar{\mathbf{r}}')\bar{\mathbf{C}}_{\text{ep}}^T\delta\ddot{\mathbf{a}}_p = \sum_{i=1}^{n_F}\mathbf{r}'_{oi}\delta\mathbf{F}_{oi} + \sum_{i=1}^{n_M}\delta\mathbf{M}_{oi} \end{aligned} \quad (B6)$$

$$m_p \bar{\Phi}^T \delta \ddot{\mathbf{R}}_{no} + (\Phi_\theta \mathbf{I}_{po} - m_p \bar{\Phi}^T \bar{\mathbf{R}}') \delta \ddot{\mathbf{a}}_o + (\mathbf{I}_{n \times n} + \Phi_\theta \mathbf{I}_{po} \Phi_\theta^T + m_p \bar{\Phi}^T \bar{\Phi}) \delta \ddot{\mathbf{q}} + (\Phi_\theta \mathbf{I}_{po} - m_p \bar{\Phi}^T \bar{\mathbf{r}}' \bar{\mathbf{C}}_{ep}^T) \delta \ddot{\mathbf{a}}_p = \sum_{i=1}^{n_F} \Phi_{uoi} \delta \mathbf{F}_{oi} + \sum_{i=1}^{n_M} \Phi_{\theta oi} \delta \mathbf{M}_{oi} \quad (B7)$$

A matrix form of equations (B1, B6, B7, and B4) may be assembled as follows:

$$\mathbf{A} \delta \ddot{\boldsymbol{\eta}} + \mathbf{B} \delta \dot{\boldsymbol{\eta}} + \mathbf{C} \delta \boldsymbol{\eta} = \mathbf{E} \delta \mathbf{u} \quad (B8)$$

Here \mathbf{A} is a (4×4) block matrix of overall dimension $(n + 9)$ square. Matrices \mathbf{B} and \mathbf{C} are essentially null except for n diagonal terms corresponding to the elastic motion. The vector $\delta \boldsymbol{\eta}$ of system parameters is $(n + 9) \times 1$, and is given by:

$$\delta \boldsymbol{\eta} = \delta(\mathbf{R}_{no}^T; \mathbf{a}_o^T; \mathbf{q}^T; \mathbf{a}_{po}^T)^T \quad (B9)$$

The transformed payload angles $\delta \mathbf{a}_{po} = \bar{\mathbf{C}}_{ep}^T \delta \mathbf{a}_p$ must appear in the vector of system parameters for symmetric equations to result. The vector of control inputs is:

$$\delta \mathbf{u} = (\delta \mathbf{F}_{o1}^T, \dots, \delta \mathbf{F}_{on_F}^T; \delta \mathbf{M}_{o1}^T, \dots, \delta \mathbf{M}_{on_M}^T; \delta \mathbf{T}_p^T)^T \quad (B10)$$

The equations for an LFSP supporting multiple payloads will be given as obtained from the linearized equations (B1, B6, B7, and B4). Here a large flexible space platform described in part by n known elastic modes, is acted on by n_F external forces and n_M moments exclusive of the torques required to point the n_p payloads.

The equations of motion for multi-payload configurations may be written in the following form:

$$\mathbf{A} \delta \ddot{\boldsymbol{\eta}} + \mathbf{B} \delta \dot{\boldsymbol{\eta}} + \mathbf{C} \delta \boldsymbol{\eta} = \mathbf{E} \delta \mathbf{u} \quad (B11)$$

The vector $\delta \boldsymbol{\eta}$ is dimensioned $(6 + n + 3n_p) \times 1$, and is given by:

$$\delta \boldsymbol{\eta} = \delta(\mathbf{R}_{no}^T; \mathbf{a}_o^T; \mathbf{q}^T; \mathbf{a}_{1o}^T, \dots, \mathbf{a}_{n_po}^T)^T \quad (B12)$$

The vector of force and moment inputs $\delta \mathbf{u}$ is dimensioned $(3n_F + 3n_M + 3n_p) \times 1$, and is written as follows:

$$\delta \mathbf{u} = \delta(\mathbf{F}_{o1}, \dots, \mathbf{F}_{on_F}; \mathbf{M}_{o1}, \dots, \mathbf{M}_{on_M}; \mathbf{T}_1, \dots, \mathbf{T}_{n_p}) \quad (B13)$$

where $\delta \mathbf{F}_{oi}$, $\delta \mathbf{M}_{oi}$, and $\delta \mathbf{T}_i$ are external LFSP forces and moments, and payload gimbal torques, respectively.

The coefficient matrices \mathbf{A} , \mathbf{B} , and \mathbf{C} , dimensioned $(6 + n + 3n_p)$ square, and \mathbf{E} , dimensioned $(6 + n + 3n_p) \times (3n_F + 3n_M + 3n_p)$, are given below. Matrix \mathbf{A} is essentially a “mass/inertia” matrix given by:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1[(6+n) \times (6+n)] & \mathbf{A}_2[(6+n) \times 3n_p] \\ \mathbf{A}_3[3n_p \times (6+n)] & \mathbf{A}_4(3n_p \times 3n_p) \end{bmatrix} \quad (B14)$$

The blocks comprising matrix \mathbf{A} are:

$$\mathbf{A}_1 = \begin{bmatrix} \left(m_o + \sum_{i=1}^{n_p} m_i\right) \mathbf{I}_{3 \times 3} & -\sum_{i=1}^{n_p} m_i \bar{\mathbf{R}}'_i & \sum_{i=1}^{n_p} m_i \bar{\Phi}_i \\ \sum_{i=1}^{n_p} m_i \bar{\mathbf{R}}'_i & \mathbf{I}_o - \sum_{i=1}^{n_p} (\mathbf{I}_{i_o} - m_i \bar{\mathbf{R}}'_i \bar{\mathbf{R}}'_i) & \sum_{i=1}^{n_p} (\mathbf{I}_{i_o} + m_i \bar{\mathbf{R}}'_i \bar{\Phi}_i) \\ \sum_{i=1}^{n_p} m_i \bar{\Phi}_i^T & -\sum_{i=1}^{n_p} (\Phi_{\theta i} \mathbf{I}_{i_o} - m_i \bar{\Phi}_i^T \bar{\mathbf{R}}'_i) & \mathbf{I}_{n \times n} + \sum_{i=1}^{n_p} \Phi_{\theta i} \mathbf{I}_{i_o} \Phi_{\theta i}^T \\ & & + \sum_{i=1}^{n_p} m_i \bar{\Phi}_i^T \bar{\Phi}_i \end{bmatrix} \quad (B15a)$$

$$\mathbf{A}_2 = \begin{bmatrix} -m_1 \bar{\mathbf{r}}'_1 & \dots & -m_{n_p} \bar{\mathbf{r}}'_{n_p} \\ \mathbf{I}_{1_o} - m_1 \bar{\mathbf{R}}'_1 \bar{\mathbf{r}}'_1 & \dots & \mathbf{I}_{n_p o} - m_{n_p} \bar{\mathbf{R}}'_{n_p} \bar{\mathbf{r}}'_{n_p} \\ \Phi_{\theta 1} \mathbf{I}_{1_o} - m_1 \bar{\Phi}_1^T \bar{\mathbf{r}}'_1 & \dots & \Phi_{\theta n_p} \mathbf{I}_{n_p o} - m_{n_p} \bar{\Phi}_{n_p}^T \bar{\mathbf{r}}'_{n_p} \end{bmatrix} \quad (B15b)$$

$$\mathbf{A}_3 = \begin{bmatrix} m_1 \bar{\mathbf{r}}'_1 & \mathbf{I}_{1_o} - m_1 \bar{\mathbf{r}}'_1 \bar{\mathbf{R}}'_1 & \mathbf{I}_{1_o} \Phi_{\theta 1}^T + m_1 \bar{\mathbf{r}}'_1 \bar{\Phi}_1 \\ \vdots & \vdots & \vdots \\ m_{n_p} \bar{\mathbf{r}}'_{n_p} & \mathbf{I}_{n_p o} - m_{n_p} \bar{\mathbf{r}}'_{n_p} \bar{\mathbf{R}}'_{n_p} & \mathbf{I}_{n_p o} \Phi_{\theta n_p}^T + m_{n_p} \bar{\mathbf{r}}'_{n_p} \bar{\Phi}_{n_p} \end{bmatrix} \quad (B15c)$$

$$\mathbf{A}_4 = \begin{bmatrix} \mathbf{I}_{1_o} - m_1 \bar{\mathbf{r}}'_1 \bar{\mathbf{r}}'_1 & \dots & \mathbf{0}_{3 \times 3} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{3 \times 3} & \dots & \mathbf{I}_{n_p o} - m_{n_p} \bar{\mathbf{r}}'_{n_p} \bar{\mathbf{r}}'_{n_p} \end{bmatrix} \quad (B15d)$$

where:

$$\bar{\mathbf{r}}'_i = (\mathbf{C}_{ei}^T \mathbf{r}_i)^' \quad \bar{\mathbf{R}}'_i = (\mathbf{R}'_{oi} + \bar{\mathbf{r}}'_i) \quad \bar{\Phi}_i = (\Phi_{ui}^T - \bar{\mathbf{r}}'_i \Phi_{\theta i}^T) \quad (B16)$$

Matrices **B** and **C** are for the most part null except for diagonal blocks multiplying the n modal coordinates. **B** and **C** are then modal damping and modal stiffness matrices written as:

$$\begin{aligned} \mathbf{B} &= \text{diag}[\mathbf{0}_{(6 \times 6)}, \mathbf{D}_{(n \times n)}, \mathbf{0}_{(3n_p \times 3n_p)}] \\ \mathbf{C} &= \text{diag}[\mathbf{0}_{(6 \times 6)}, \mathbf{\Lambda}_{(n \times n)}, \mathbf{0}_{(3n_p \times 3n_p)}] \end{aligned} \quad (B17)$$

where:

$$\begin{aligned} \mathbf{D} &= [\text{diag}(2\zeta_i \omega_i)]_{(n \times n)} \\ \mathbf{\Lambda} &= [\text{diag}(\omega_i^2)]_{(n \times n)} \end{aligned} \quad (B18)$$

The coefficient matrix **E** is written as:

$$\mathbf{E} = \begin{bmatrix} \mathbf{E}_1[(6+n) \times 3n_F] & \mathbf{E}_2[(6+n) \times 3n_M] & \mathbf{0}_{[(6+n) \times 3n_p]} \\ \mathbf{0}_{(3n_p \times 3n_F)} & \mathbf{0}_{(3n_p \times 3n_M)} & \mathbf{I}_{(3n_p \times 3n_p)} \end{bmatrix} \quad (B19)$$

where **E**₁ and **E**₂ are given by:

$$\mathbf{E}_1 = \begin{bmatrix} \mathbf{I}_{3 \times 3} & \dots & \mathbf{I}_{3 \times 3} \\ \mathbf{r}'_{o1} & \dots & \mathbf{r}'_{on_F} \\ \Phi_{uo1} & \dots & \Phi_{uon_F} \end{bmatrix}, \quad \mathbf{E}_2 = \begin{bmatrix} \mathbf{0}_{3 \times 3} & \dots & \mathbf{0}_{3 \times 3} \\ \mathbf{I}_{3 \times 3} & \dots & \mathbf{I}_{3 \times 3} \\ \Phi_{\theta o1} & \dots & \Phi_{\theta on_M} \end{bmatrix} \quad (B20)$$